Baltic Way ’11
Greifswald, Germany

Problems –English version–

Problem 1 The real numbers $x_1, \dots, x_{2011}$ satisfy

$$x_1 + x_2 = 2x'_1, \quad x_2 + x_3 = 2x'_2, \quad \dots, \quad x_{2011} + x_1 = 2x'_{2011}$$

where $x'_1, x'_2, \dots, x'_{2011}$ is a permutation of $x_1, x_2, \dots, x_{2011}$. Prove that $x_1 = x_2 = \cdots = x_{2011}$.

Problem 2 Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function such that, for all integers x and y, the following holds:

$$f(f(x) - y) = f(y) - f(f(x)).$$

Show that f is bounded, i.e. that there is a constant C such that

$$-C < f(x) < C$$

for all integers x.

Problem 3 A sequence $a_1, a_2, a_3, \dots$ of non-negative integers is such that a_{n+1} is the last digit of $a_n^n + a_{n-1}$ for all $n > 2$. Is it always true that for some n_0 the sequence $a_{n_0}, a_{n_0+1}, a_{n_0+2}, \dots$ is periodic?

Problem 4 Let a, b, c, d be non-negative reals such that $a + b + c + d = 4$. Prove the inequality

$$\frac{a}{a^3+8} + \frac{b}{b^3+8} + \frac{c}{c^3+8} + \frac{d}{d^3+8} \le \frac{4}{9}.$$

Problem 5 Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(f(x)) = x^2 - x + 1$$

for all real numbers x. Determine $f(0)$.

Problem 6 Let n be a positive integer. Prove that the number of lines which go through the origin and precisely one other point with integer coordinates (x, y), $0 \le x, y \le n$, is at least $\frac{n^2}{4}$.

Problem 7 Let T denote the 15-element set $\{10a + b : a, b \in \mathbb{Z}, 1 \le a < b \le 6\}$. Let S be a subset of T in which all six digits $1, 2, \dots, 6$ appear and in which no three elements together use all these six digits. Determine the largest possible size of S.

Problem 8 In Greifswald there are three schools called A, B and C, each of which is attended by at least one student. Among any three students, one from A, one from B and one from C, there are two knowing each other and two not knowing each other. Prove that at least one of the following holds:

- Some student from A knows all students from B.
- Some student from B knows all students from C.
- Some student from C knows all students from A.

Problems –English version–

Problem 9 Given a rectangular grid, split into $m \times n$ squares, a colouring of the squares in two colours (black and white) is called *valid* if it satisfies the following conditions:

- All squares touching the border of the grid are coloured black.
- No four squares forming a 2×2-square are coloured in the same colour.
- No four squares forming a 2×2-square are coloured in such a way that only diagonally touching squares have the same colour.

Which grid sizes $m \times n$ (with $m, n \geq 3$) have a valid colouring?

Problem 10 Two persons play the following game with integers. The initial number is 2011^{2011}. The players move in turns. Each move consists of subtraction of an integer between 1 and 2010 inclusive, or division by 2011, rounding down to the closest integer when necessary. The player who first obtains a non-positive integer wins. Which player has a winning strategy?

Problem 11 Let AB and CD be two diameters of the circle $\mathcal{C}$. For an arbitrary point P on $\mathcal{C}$, let R and S be the feet of the perpendiculars from P to AB and CD, respectively. Show that the length of RS is independent of the choice of P.

Problem 12 Let P be a point inside a square $ABCD$ such that $PA : PB : PC$ is $1 : 2 : 3$. Determine the angle $\angle BPA$.

Problem 13 Let E be an interior point of the convex quadrilateral $ABCD$. Construct triangles $\triangle ABF$, $\triangle BCG$, $\triangle CDH$ and $\triangle DAI$ on the outside of the quadrilateral such that the similarities $\triangle ABF \sim \triangle DCE$, $\triangle BCG \sim \triangle ADE$, $\triangle CDH \sim \triangle BAE$ and $\triangle DAI \sim \triangle CBE$ hold. Let P, Q, R and S be the projections of E on the lines AB, BC, CD and DA, respectively. Prove that if the quadrilateral $PQRS$ is cyclic, then

$$EF \cdot CD = EG \cdot DA = EH \cdot AB = EI \cdot BC\,.$$

Problem 14 The incircle of a triangle ABC touches the sides BC, CA, AB at D, E, F, respectively. Let G be a point on the incircle such that FG is a diameter. The lines EG and FD intersect at H. Prove that $CH \parallel AB$.

Problem 15 Let $ABCD$ be a convex quadrilateral such that $\angle ADB = \angle BDC$. Suppose that a point E on the side AD satisfies the equality

$$AE \cdot ED + BE^2 = CD \cdot AE.$$

Show that $\angle EBA = \angle DCB$.

Problem 16 Let a be any integer. Define the sequence $x_0, x_1, \ldots$ by $x_0 = a$, $x_1 = 3$ and

$$x_n = 2x_{n-1} - 4x_{n-2} + 3 \text{ for all } n > 1.$$

Determine the largest integer k_a for which there exists a prime p such that p^{k_a} divides $x_{2011} - 1$.

Problem 17 Determine all positive integers d such that whenever d divides a positive integer n, d will also divide any integer obtained by rearranging the digits of n.

Problem 18 Determine all pairs (p, q) of primes for which both $p^2 + q^3$ and $q^2 + p^3$ are perfect squares.

Problem 19 Let $p \neq 3$ be a prime number. Show that there is a non-constant arithmetic sequence of positive integers $x_1, x_2, \ldots, x_p$ such that the product of the terms of the sequence is a cube.

Problem 20 An integer $n \geq 1$ is called *balanced* if it has an even number of distinct prime divisors. Prove that there exist infinitely many positive integers n such that there are exactly two balanced numbers among n, $n+1$, $n+2$ and $n+3$.