

## Problems –English version–

**Problem 1** The real numbers  $x_1, \dots, x_{2011}$  satisfy

$$x_1 + x_2 = 2x'_1, \quad x_2 + x_3 = 2x'_2, \quad \dots, \quad x_{2011} + x_1 = 2x'_{2011}$$

where  $x'_1, x'_2, \dots, x'_{2011}$  is a permutation of  $x_1, x_2, \dots, x_{2011}$ . Prove that  $x_1 = x_2 = \dots = x_{2011}$ .

**Problem 2** Let  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  be a function such that, for all integers  $x$  and  $y$ , the following holds:

$$f(f(x) - y) = f(y) - f(f(x)).$$

Show that  $f$  is bounded, i.e. that there is a constant  $C$  such that

$$-C < f(x) < C$$

for all integers  $x$ .

**Problem 3** A sequence  $a_1, a_2, a_3, \dots$  of non-negative integers is such that  $a_{n+1}$  is the last digit of  $a_n^n + a_{n-1}$  for all  $n > 2$ . Is it always true that for some  $n_0$  the sequence  $a_{n_0}, a_{n_0+1}, a_{n_0+2}, \dots$  is periodic?

**Problem 4** Let  $a, b, c, d$  be non-negative reals such that  $a + b + c + d = 4$ . Prove the inequality

$$\frac{a}{a^3 + 8} + \frac{b}{b^3 + 8} + \frac{c}{c^3 + 8} + \frac{d}{d^3 + 8} \leq \frac{4}{9}.$$

**Problem 5** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function such that

$$f(f(x)) = x^2 - x + 1$$

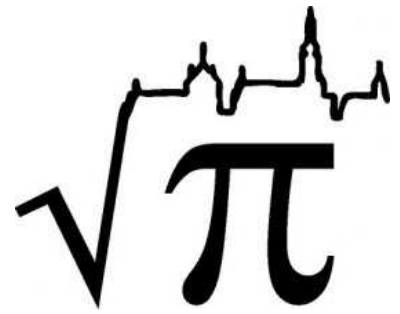
for all real numbers  $x$ . Determine  $f(0)$ .

**Problem 6** Let  $n$  be a positive integer. Prove that the number of lines which go through the origin and precisely one other point with integer coordinates  $(x, y)$ ,  $0 \leq x, y \leq n$ , is at least  $\frac{n^2}{4}$ .

**Problem 7** Let  $T$  denote the 15-element set  $\{10a + b : a, b \in \mathbb{Z}, 1 \leq a < b \leq 6\}$ . Let  $S$  be a subset of  $T$  in which all six digits  $1, 2, \dots, 6$  appear and in which no three elements together use all these six digits. Determine the largest possible size of  $S$ .

**Problem 8** In Greifswald there are three schools called  $A, B$  and  $C$ , each of which is attended by at least one student. Among any three students, one from  $A$ , one from  $B$  and one from  $C$ , there are two knowing each other and two not knowing each other. Prove that at least one of the following holds:

- Some student from  $A$  knows all students from  $B$ .
- Some student from  $B$  knows all students from  $C$ .
- Some student from  $C$  knows all students from  $A$ .



## Problems –English version–

**Problem 9** Given a rectangular grid, split into  $m \times n$  squares, a colouring of the squares in two colours (black and white) is called *valid* if it satisfies the following conditions:

- All squares touching the border of the grid are coloured black.
- No four squares forming a  $2 \times 2$ -square are coloured in the same colour.
- No four squares forming a  $2 \times 2$ -square are coloured in such a way that only diagonally touching squares have the same colour.

Which grid sizes  $m \times n$  (with  $m, n \geq 3$ ) have a valid colouring?

**Problem 10** Two persons play the following game with integers. The initial number is  $2011^{2011}$ . The players move in turns. Each move consists of subtraction of an integer between 1 and 2010 inclusive, or division by 2011, rounding down to the closest integer when necessary. The player who first obtains a non-positive integer wins. Which player has a winning strategy?

**Problem 11** Let  $AB$  and  $CD$  be two diameters of the circle  $\mathcal{C}$ . For an arbitrary point  $P$  on  $\mathcal{C}$ , let  $R$  and  $S$  be the feet of the perpendiculars from  $P$  to  $AB$  and  $CD$ , respectively. Show that the length of  $RS$  is independent of the choice of  $P$ .

**Problem 12** Let  $P$  be a point inside a square  $ABCD$  such that  $PA : PB : PC$  is  $1 : 2 : 3$ . Determine the angle  $\angle BPA$ .

**Problem 13** Let  $E$  be an interior point of the convex quadrilateral  $ABCD$ . Construct triangles  $\triangle ABF$ ,  $\triangle BCG$ ,  $\triangle CDH$  and  $\triangle DAI$  on the outside of the quadrilateral such that the similarities  $\triangle ABF \sim \triangle DCE$ ,  $\triangle BCG \sim \triangle ADE$ ,  $\triangle CDH \sim \triangle BAE$  and  $\triangle DAI \sim \triangle CBE$  hold. Let  $P$ ,  $Q$ ,  $R$  and  $S$  be the projections of  $E$  on the lines  $AB$ ,  $BC$ ,  $CD$  and  $DA$ , respectively. Prove that if the quadrilateral  $PQRS$  is cyclic, then

$$EF \cdot CD = EG \cdot DA = EH \cdot AB = EI \cdot BC.$$

**Problem 14** The incircle of a triangle  $ABC$  touches the sides  $BC$ ,  $CA$ ,  $AB$  at  $D$ ,  $E$ ,  $F$ , respectively. Let  $G$  be a point on the incircle such that  $FG$  is a diameter. The lines  $EG$  and  $FD$  intersect at  $H$ . Prove that  $CH \parallel AB$ .

**Problem 15** Let  $ABCD$  be a convex quadrilateral such that  $\angle ADB = \angle BDC$ . Suppose that a point  $E$  on the side  $AD$  satisfies the equality

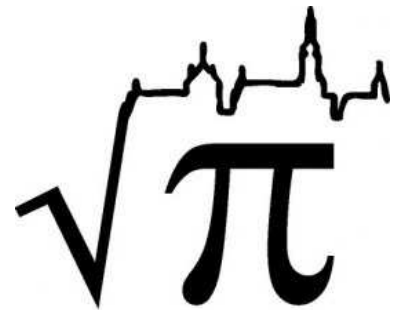
$$AE \cdot ED + BE^2 = CD \cdot AE.$$

Show that  $\angle EBA = \angle DCB$ .

**Problem 16** Let  $a$  be any integer. Define the sequence  $x_0, x_1, \dots$  by  $x_0 = a$ ,  $x_1 = 3$  and

$$x_n = 2x_{n-1} - 4x_{n-2} + 3 \text{ for all } n > 1.$$

Determine the largest integer  $k_a$  for which there exists a prime  $p$  such that  $p^{k_a}$  divides  $x_{2011} - 1$ .



## Problems –English version–

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**Problem 17** Determine all positive integers  $d$  such that whenever  $d$  divides a positive integer  $n$ ,  $d$  will also divide any integer obtained by rearranging the digits of  $n$ .

**Problem 18** Determine all pairs  $(p, q)$  of primes for which both  $p^2 + q^3$  and  $q^2 + p^3$  are perfect squares.

**Problem 19** Let  $p \neq 3$  be a prime number. Show that there is a non-constant arithmetic sequence of positive integers  $x_1, x_2, \dots, x_p$  such that the product of the terms of the sequence is a cube.

**Problem 20** An integer  $n \geq 1$  is called *balanced* if it has an even number of distinct prime divisors. Prove that there exist infinitely many positive integers  $n$  such that there are exactly two balanced numbers among  $n, n + 1, n + 2$  and  $n + 3$ .