Baltic Way 2017

Sorø, November 11th, 2017

Problems and solutions

Problem 1. Let $a_0, a_1, a_2, \ldots$ be an infinite sequence of real numbers satisfying $\frac{a_{n-1}+a_{n+1}}{2} \geq a_n$ for all positive integers n. Show that

$$\frac{a_0 + a_{n+1}}{2} \geq \frac{a_1 + a_2 + \ldots + a_n}{n}$$

holds for all positive integers n.

Solution

From the inequality $\frac{a_{n-1}+a_{n+1}}{2} \geq a_n$ we get $a_{n+1} - a_n \geq a_n - a_{n-1}$. Inductively this yields that $a_{l+1} - a_l \geq a_{k+1} - a_k$ for any positive integers $l > k$, which rewrites as

$$a_{l+1} + a_k \geq a_l + a_{k+1}$$

Now fix n and define $b_m = a_m + a_{n+1-m}$ for $m = 0, \ldots n+1$. For $m < \frac{n}{2}$, we can apply the above for $(l, k) = (n-m, m)$ yielding

$$b_m = a_{n+1-m} + a_m \geq a_{n-m} + a_{m+1} = b_{m+1}$$

Also by symmetry $b_m = b_{n+1-m}$. Thus

$$b_0 = \max_{m=0,\ldots,n+1} b_m \geq \max_{m=1,\ldots,n} b_m \geq \frac{b_1 + \cdots + b_n}{n}$$

substituting back yields the desired inequality.

Problem 2. Does there exist a finite set of real numbers such that their sum equals 2, the sum of their squares equals 3, the sum of their cubes equals 4, …, and the sum of their ninth powers equals 10?

Solution

Answer: no.

Assume that such a set of numbers $\{a_1, \ldots, a_n\}$ exists. Summing up the inequalities

$$2a_i^3 \leq a_i^2 + a_i^4$$

for all i we obtain the inequality $8 \leq 8$. Therefore all the inequalities are in fact equalities. This is possible for the cases $a_i = 0$ or $a_i = 1$ only, but the elements of a set are all different.

(Remark: Even if the a_i's are allowed to be equal it is clear that only 0's or only 1's do not satisfy the problem conditions.)

Problem 3. Positive integers $x_1, \ldots, x_m$ (not necessarily distinct) are written on a blackboard. It is known that each of the numbers $F_1, \ldots, F_{2018}$ can be represented as a sum of one or more of the numbers on the blackboard. What is the smallest possible value of m?

(Here $F_1, \ldots, F_{2018}$ are the first 2018 Fibonacci numbers: $F_1 = F_2 = 1$, $F_{k+1} = F_k + F_{k-1}$ for $k > 1$.)

Solution

Answer: the minimal value for m is 1009.

Construction: Define $x_i = F_{2i-1}$. This works since $F_{2k} = F_1 + F_3 + \ldots + F_{2k-1}$, for all k, which can easily be proved by induction. Minimality: Again by induction we get that $F_{k+2} = 1 + F_1 + F_2 + \ldots + F_k$, for all k, which means that

$$F_{k+2} > F_1 + F_2 + \ldots + F_k. \qquad (*)$$

Consider the numbers that have been used for the representing the first k Fibonacci numbers. Then the sum of these x_i's is less than F_{k+2} due to $(*)$. Thus, at least one additional number is required to deal with F_{k+2}. This establishes the lower bound $m \leq 1009$.

Problem 4. A *linear form* in k variables is an expression of the form $P(x_1, \ldots, x_k) = a_1 x_1 + \ldots + a_k x_k$ with real constants $a_1, \ldots, a_k$. Prove that there exist a positive integer n and linear forms $P_1, \ldots, P_n$ in 2017 variables such that the equation

$$x_1 \cdot x_2 \cdot \ldots \cdot x_{2017} = P_1(x_1, \ldots, x_{2017})^{2017} + \ldots + P_n(x_1, \ldots, x_{2017})^{2017}$$

holds for all real numbers $x_1, \ldots, x_{2017}$.

Solution

Solution 1: For every $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{\pm 1\}^{2017}$ let

$$P_\varepsilon(X_1, \dots, X_{2017}) = \varepsilon_1 X_1 + \dots + \varepsilon_{2017} X_{2017}$$

and $\beta_\varepsilon = \varepsilon_1 \cdots \varepsilon_{2017}$. Consider

$$\begin{aligned} g(X_1, \dots, X_k) &:= \sum_\varepsilon (\beta_\varepsilon P_\varepsilon(X_1, \dots, X_{2017}))^{2017} \\ &= \sum_{\varepsilon_1, \dots, \varepsilon_{2017}} \varepsilon_1 \cdots \varepsilon_{2017} (\varepsilon_1 X_1 + \dots + \varepsilon_{2017} X_{2017})^{2017}. \end{aligned}$$

If we choose $X_1 = 0$, then every combination $(\varepsilon_2 X_2 + \dots + \varepsilon_{2017} X_{2017})^{2017}$ occurs exactly twice and with opposite signs in the above sum. Hence, $g(0, X_2, \dots, X_{2017}) \equiv 0$. The analogous statements are true for all other variables. Consequently, g is divisible by $X_1 \dots X_{2017}$, and thereby of the form $c X_1 \dots X_{2017}$ for some real constant c. If $c \neq 0$, then both sides can be divided by c, and we obtain a representation with $n = 2^{2017}$ linear forms.

With $X_1 = \dots = X_{2017} = 1$ we get

$$\begin{aligned} c &= \sum_\varepsilon \varepsilon_1 \cdot \ldots \cdot \varepsilon_{2017} (\varepsilon_1 + \ldots + \varepsilon_{2017})^{2017} \\ &= \sum_\varepsilon \sum_{\substack{k_1, \dots, k_{2017} \\ k_1 + \ldots + k_{2017} = 2017}} \binom{2017}{k_1, \dots, k_{2017}} \varepsilon_1^{k_1+1} \cdot \ldots \cdot \varepsilon_{2017}^{k_{2017}+1} \end{aligned}$$

The part of the sum with k_1 even is zero since

$$\sum_{\substack{k_1 \text{ even}, \dots, k_{2017} \\ k_1 + \ldots + k_{2017} = 2017}} \binom{2017}{k_1, \dots, k_{2017}} \left(\sum_{\varepsilon, \varepsilon_1 = 1} \varepsilon_2^{k_2+1} \cdot \ldots \cdot \varepsilon_{2017}^{k_{2017}+1} + \sum_{\varepsilon, \varepsilon_1 = -1} (-1)^{k_1+1} \cdot \ldots \cdot \varepsilon_{2017}^{k_{2017}+1} \right) = 0$$

Now we may consider the part of the sum with k_1 odd. Similarly the part of this new sum with k_2 even equals 0. Doing this for all the variables we get

$$c = \sum_\varepsilon \sum_{\substack{k_1 \text{ odd}, \dots, k_{2017} \text{ odd} \\ k_1 + \ldots + k_{2017} = 2017}} \binom{2017}{k_1, \dots, k_{2017}} = 2^{2017} \binom{2017}{1, \dots, 1} = 2^{2017} \cdot 2017! \neq 0$$

Finally, we can even merge the two forms with opposite choices of the signs to obtain a representation with 2^{2016} linear forms.

Solution 2: We show by induction that for every integer $k \geq 1$ there exist an $n = n_k$, real numbers $\lambda_1, \dots, \lambda_{n_k}$ and linear forms $P_{k,1}, \dots, P_{k,n_k}$ in k variables such that

$$x_1 \dots x_k = \lambda_1 P_{k,1}(x_1, \dots, x_k)^k + \dots + \lambda_{n_k} P_{k,n_k}(x_1, \dots, x_k)^k.$$

For $k = 1$ we can choose $n = n_1 = 1$ and $P_{1,1}(x_1) = x_1$. Now for the induction step, we observe that

$$x_1 \dots x_k y = \lambda_1 P_{k,1}(x_1, \dots, x_k)^k y + \dots + \lambda_k P_{k,n_k}(x_1, \dots, x_k)^k y$$

Thus it suffices to write $X^k Y$ as a linear combination of $(k+1)$-th powers of linear forms in X and Y. The set-up

$$X^k Y = \sum_{i=1}^m \alpha_i (X + \beta_i Y)^{k+1}$$

leads to the equations $\sum_i \alpha_i \beta_i = \frac{1}{k+1}$ and $\sum_i \alpha_i \beta_i^d = 0$ for $d = 0, 2, 3, \dots, k+1$. Choosing $m = k+2$ and distinct values for the β_i's, this becomes a system of $k+2$ linear equations in the $k+2$ variables α_i. If the system had no solution, then the left-hand sides of the equations would be linearly dependent. On the other hand, given c_j, $j = 0, 1, \dots, k+1$ with $\sum_j c_j \beta_i^j = 0$ for all i, the polynomial $P(x) = \sum_j c_j x^j$ has degree at most $k+1$ and the $k+2$ distinct zeros $b_1, \dots, b_{k+2}$ and, hence, is the zero polynomial. Consequently, the system has a solution, and we can choose $n_{k+1} = (k+2) n_k$ and the induction is complete.

The above gives us

$$x_1 \dots x_{2017} = \lambda_1 P_{2017,1}(x_1,\dots,x_{2017})^{2017} + \dots + \lambda_{n_{2017}} P_{2017,n_{2017}}(x_1,\dots,x_{2017})^{2017}$$
$$= \left(\lambda_1^{1/2017} P_{2017,1}(x_1,\dots,x_{2017})\right)^{2017} + \dots + \left(\lambda_{n_{2017}}^{1/2017} P_{2017,n_{2017}}(x_1,\dots,x_{2017})\right)^{2017},$$

as wanted.

Remark: Of course, the consistency of the system of linear equations also follows by the fact that the determinant of the coefficient matrix does not vanish as it is of Vandermonde's type.

Problem 5. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x^2y) = f(xy) + yf(f(x)+y)$$

for all real numbers x and y.

Solution
Answer: $f(y)=0$.

By substituting $x=0$ into the original equation we obtain $f(0)=f(0)+yf(f(0)+y)$, which after simplifying yields $yf(f(0)+y)=0$. This has to hold for every y.

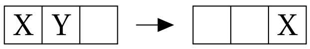
Now let's substitute $y=-f(0)$. We get that $-(f(0))^2=0$, which gives us $f(0)=0$.

By plugging the last result into $yf(f(0)+y)=0$ we now get that $yf(y)=0$.

Therefore if $y\neq 0$ then $f(y)=0$.

Altogether we have shown that $f(y)=0$ for every y is the only possible solution, and it clearly is a solution.

Problem 6. Fifteen stones are placed on a 4×4 board, one in each cell, the remaining cell being empty. Whenever two stones are on neighbouring cells (having a common side), one may jump over the other to the opposite neighbouring cell, provided this cell is empty. The stone jumped over is removed from the board.



For which initial positions of the empty cell is it possible to end up with exactly one stone on the board?

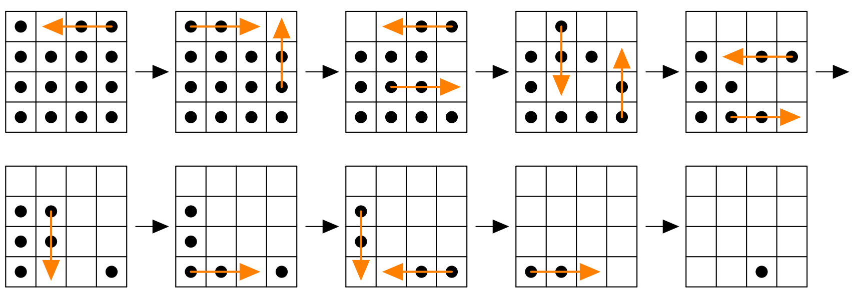
Solution
There are three types of cells on the board: corner cells, edge cells and centre cells. Colour the cells in three distinct colours as follows.

A	B	C	A
B	C	A	B
C	A	B	C
A	B	C	A

Suppose there are initially a, b, c stones on cells of colours A, B, C, respectively. With each move, one of these numbers will increase by 1, while the other two will decrease by 1. Because there are fourteen moves altogether, the game must end with a, b, c of the same parity as they originally had. There are 6, 5, 5 cells of each colour on the board, so if the game should end with a single stone remaining, the game must begin with

$$a=6, b=5, c=4 \quad \text{or} \quad a=6, b=4, c=5.$$

The empty slot should thus have colour B or C. This excludes the corner cells and two of the centre cells. However, by symmetry (changing the colouring), the two remaining centre cells will also be excluded. Hence the empty space at the beginning must be at an edge cell. That the game is indeed winnable in this case can be seen from the sequence of moves here:



Problem 7. Each edge of a complete graph on 30 vertices is coloured either red or blue. It is allowed to choose a non-monochromatic triangle and change the colour of the two edges of the same colour to make the triangle monochromatic. Prove that by using this operation repeatedly it is possible to make the entire graph monochromatic.

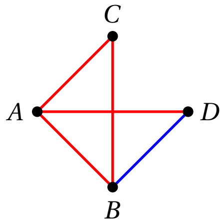
(A complete graph is a graph where any two vertices are connected by an edge.)

Solution

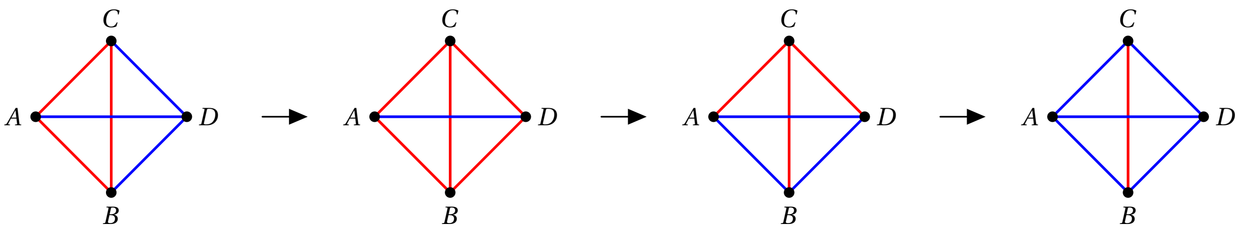
The total number of edges is odd. Assume without loss of generality that the number of blue edges is odd, and the number of red edges is even. It is clear that the parity of the number of edges of each colour does not change by the operations.

Consider a graph with maximal number of blue edges that can be obtained by these operations. Suppose that not all of its edges are blue. Then it contains at least two red edges. Because of maximality, it is not possible to have a triangle with exactæy two red edges.

Case 1. It contains two red edges AB and BC sharing a common vertex. Then edge AC is coloured in red, too. If there exists a vertex D such that the edges DA, DB, DC are not of the same colour, then wlog we can assume that DA is red and DB is blue, but then we have a triangle ABD with exactly two red edges, a contradiction.

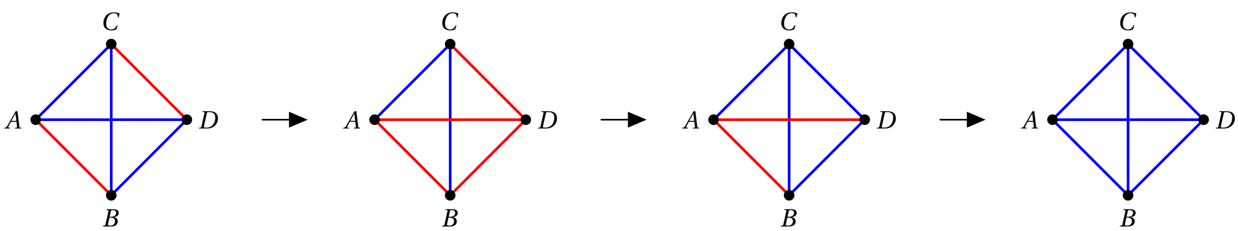


If some vertex D is connected to A, B and C with blue edges, then perform the operation on the triangles BCD, ABD, ACD, and the number of blue edges increases, a contradiction.

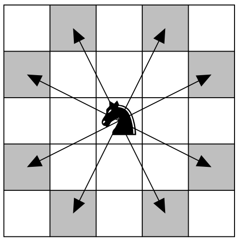


Otherwise all the vertices are connected to A, B and C with red edges. Due to parity we have at least one blue edge. If X and Y are connected by a blue edge, then perform the operation on AXY, and the number of blue edges increases, a contradiction.

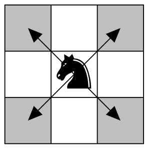
Case 2. Every two red edges have no common vertex. Let AB and CD be red edges. Perform the operation in the triangles ABD, BCD, ABD. The number of blue edges increases.



Problem 8. A chess knight has injured his leg and is limping. He alternates between a normal move and a short move where he moves to any diagonally neighbouring cell.



Normal move



Short move

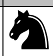
The limping knight moves on a 5×6 cell chessboard starting with a normal move. What is the largest number of moves he can make if he is starting from a cell of his own choice and is not allowed to visit any cell (including the initial cell) more than once?

Solution

Answer: 25 *moves.*

Let us enumerate the rows of the chessboard with numbers 1 to 5. We will consider only the short moves. Each short move connects two cells from rows of different parity and no two short moves has a common cell. Therefore there can be at most 12 short moves as there are just 12 cells in the rows of even parity (second and fourth). It means that the maximal number of moves is 12 short + 13 normal = 25 moves.

The figure shows that 25 moves indeed can be made.

19	5	7	9	11	
4	18	20	6	8	10
			21	26	12
17	3	24	15	13	22
2	16	14	25	23	

Problem 9. A positive integer n is *Danish* if a regular hexagon can be partitioned into n congruent polygons. Prove that there are infinitely many positive integers n such that both n and $2^n + n$ are Danish.

Solution

At first we note that $n = 3k$ is danish for any positive integer k, because a hexagon can be cut in 3 equal parallelograms each of which can afterwards be cut in k equal parallelograms (Fig 1).

Furthermore a hexagon can be cut into two equal trapezoids (Fig. 2) each of which can afterwards be cut into 4 equal trapezoids of the same shape (Fig. 3) and so on. Therefore any number of the form $n = 2 \cdot 4^k$ is also danish.

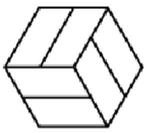


Figure 1

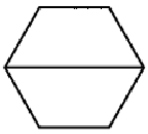


Figure 2

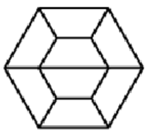


Figure 3

If we take any danish number $n = 2 \cdot 4^k$ of the second type, then

$$2^n + n = 2^{2 \cdot 4^k} + 2 \cdot 4^k \equiv 1 + 2 \equiv 0 \pmod 3$$

showing that $2^n + n$ is also a danish number.

Problem 10. Maker and Breaker are building a wall. Maker has a supply of green cubical building blocks, and Breaker has a supply of red ones, all of the same size. On the ground, a row of m squares has been marked in chalk as place-holders. Maker and Breaker now take turns in placing a block either directly on one of these squares, or on top of another block already in place, in such a way that the height of each column never exceeds n. Maker places the first block.

Maker bets that he can form a green row, i.e. all m blocks at a certain height are green. Breaker bets that he can prevent Maker from achieving this. Determine all pairs (m, n) of positive integers for which Maker can make sure he wins the bet.

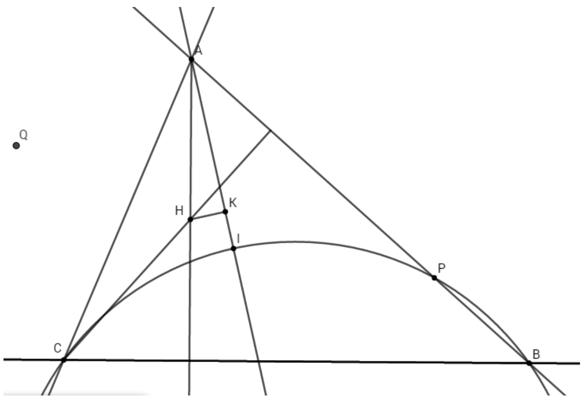
Solution
Answer: Maker has a winning strategy if $m > 1$ and $n > 1$ are both odd, or if $m = 1$.

Let us refer to the positions of the blocks in the wall by coordinates (x, y) where $x \in \{0, 1, \ldots, m-1\}$ refers to the column and $y \in \{0, 1, \ldots, n-1\}$ to the height of the block. Consider the different cases according to the parity of the parameters. In addition, there are some exceptional trivial cases.

1. If $m = 1$, then Maker trivially wins by the first move.
2. If $m > 1$, but $n = 1$, then Breaker obviously can break the only row.
3. Suppose m is even. Then Breaker has a defensive strategy based on the horizontal reflection, i.e. if Maker places a block in (x, y), then Breaker places a block in $(m-1-x, y)$. Note that this move is always available for Breaker, because m is even. It is clear that this reflection strategy breaks all the green rows.
4. Suppose then n is even but $m > 1$ is odd. Then Breaker has the same defensive stra-tegy as above based on the horizontal reflection, with one modification: Whenever Maker places a block in the middle column, Breaker does too. Since n is even, this middle column does not influence the rest of the construction. Hence this reflection strategy breaks all the green rows as above.
5. Suppose both m and n are odd, $m > 1$ and $n > 1$. Maker's strategy is the following: Maker starts with $(0, 0)$. Maker pairs the positions $(2i-1, 0)$ and $(2i, 0)$, $i = 1, 2, \ldots, \frac{m-1}{2}$, so that if Breaker places a red block in one of the positions, Maker places a green block in the other position; otherwise Maker does not use the bottom row. Maker's reply to Breaker's (x, y) with $y > 0$ is $(x, y+1)$. This strategy builds a green row at the height 2.

Problem 11. Let H and I be the orthocentre and incentre, respectively, of an acute angled triangle ABC. The circumcircle of the triangle BCI intersects the segment AB at the point P different from B. Let K be the projection of H onto AI and Q the reflection of P in K. Show that B, H and Q are collinear.

Solution
Solution 1: Let H' be the reflection of H in K. The reflection about the point K sends Q to P, and the line BH to the line through H' and orthogonal to AC. The reflection about the line AI sends P to C, and the line through H' orthogonal to AC to the line through H orthogonal to AB, but this is just BH. Since composition of the two reflections sends B, H and Q to the same line, it follows that B, H and Q are collinear.



Solution 2: Let $\alpha = \frac{1}{2}\angle BAC$, $\beta = \frac{1}{2}\angle CBA$, and $\gamma = \frac{1}{2}\angle ACB$. Clearly then $\alpha + \beta + \gamma = 90°$, which yields $\angle BIC = 180° - \beta - \gamma = 90° + \alpha$. From this we get $\angle CPA = 180° - \angle BPC = 180° - \angle BIC = 90° - \alpha$, so triangle APC is isosceles.

Now since AI is the anglebisector of $\angle PAC$ it must also be the perpendicular bisector of CP. Hence $CK = PK = QK$ so triangle KCQ is isosceles. Additionally AI bisects PQ so AI is the midline of triangle PCQ parallel to CQ. Since KH is perpendicular to AI, it is also perpendicular to CQ, so we may then conclude by symmetry that HCQ is also isosceles. Moreover $\angle QCA = \angle IAC = \alpha$, and $\angle ACH = 90° - 2\alpha$, so $\angle QCH = \alpha + 90° - 2\alpha = 90° - \alpha$, which means that triangles HCQ and APC are similar. In particular we have $\angle CHQ = 2\alpha$. Since also

$$180° - \angle BHC = \angle HCB + \angle CBH = 90° - 2\beta + 90° - 2\gamma = 2\alpha$$

it follows that B, H, and Q are collinear.

To prove that Q always lies outside of triangle ABC one could do the following: Since P lies on AB, angle C is larger than angle B in triangle ABC. Thus angle ADB is obtuse, where D is the intersection point between AI and BC. As QC and AI are parallel, angle QCB is obtuse. Thus Q lies outside of triangle ABC.

Problem 12. Line ℓ touches circle S_1 in the point X and circle S_2 in the point Y. We draw a line m which is parallel to ℓ and intersects S_1 in a point P and S_2 in a point Q. Prove that the ratio XP/YQ does not depend on the choice of m.

Solution
Let T be the second intersection point of PQ and S_1 and R be the second intersection point of PQ and S_2. Let $\angle PXT = \alpha$, $\angle RYQ = \beta$. It is evident that $PX = XT$, $RY = YQ$. Calculate the ratio $\text{area}_{PXT}/\text{area}_{RYQ}$ by two different ways. First,

$$\frac{\text{area}_{PXT}}{\text{area}_{RYQ}} = \frac{XP^2 \sin\alpha}{YQ^2 \sin\beta}.$$

Second,

$$\frac{\text{area}_{PXT}}{\text{area}_{RYQ}} = \frac{PT}{RQ} = \frac{2R_1 \sin\alpha}{2R_2 \sin\beta}.$$

Equating these expressions we obtain

$$\frac{XP}{YQ} = \sqrt{\frac{R_1}{R_2}}.$$

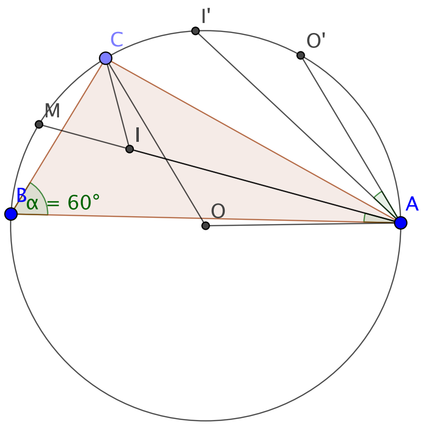
Problem 13. Let ABC be a triangle in which $\angle ABC = 60°$. Let I and O be the incentre and circumcentre of ABC, respectively. Let M be the midpoint of the arc BC of the circumcircle of ABC, which does not contain the point A. Determine $\angle BAC$ given that $MB = OI$.

Solution
Since $\angle ABC = 60°$, we have $\angle AIC = \angle AOC = 120°$. Let I', O' be the points symmetric to I, O with respect to AC, respectively. Then I' and O' lie on the circumcircle of ABC. Since $O'I' = OI = MB$, the angles determined by arcs $O'I'$ and MB are equal. It follows that $\angle MAB = \angle I'AO'$.

Now, denoting $\angle BAC = \alpha$, we have

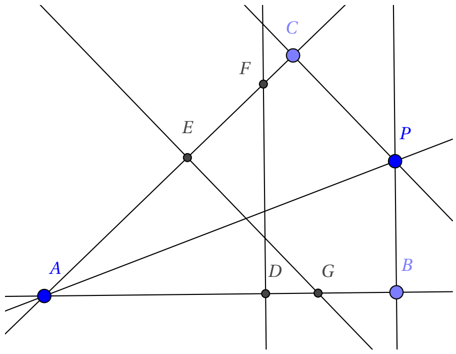
$$\angle I'AO' = \angle IAO = |\angle IAC - \angle OAC| = \left|\frac{\alpha}{2} - 30°\right|$$

It follows that $\frac{\alpha}{2} = \angle MAB = \angle I'AO' = \left|\frac{\alpha}{2} - 30°\right|$, i.e. $\alpha = 30°$.



Problem 14. Let P be a point inside the acute angle $\angle BAC$. Suppose that $\angle ABP = \angle ACP = 90°$. The points D and E are on the segments BA and CA, respectively, such that $BD = BP$ and $CP = CE$. The points F and G are on the segments AC and AB, respectively, such that DF is perpendicular to AB and EG is perpendicular to AC. Show that $PF = PG$.

Solution
As $\triangle PBD$ is an isosceles right triangle $\angle GDP = \angle BDP = 45°$. Similarly $\angle PEC = 45°$, and thus $\angle PEG = 45°$. Therefore $PGDE$ is cyclic. As $\angle GDF$ and $\angle GEF$ are right $EFGD$ is cyclic. Therefore $DGPFE$ is a cyclic pentagon. Therefore $\angle GFP = \angle GEP = 45°$. Similarly $\angle FGP = 45°$. Therefore $\triangle FPG$ is a (right) isosceles triangle.



Remark: It can be shown given two intersecting lines l and m, not perpendicular to one another and an point P. there exist unique points F and G on l and m respectively such that $\triangle FPG$ is an right isosceles triangle using similar constructions to above.

Problem 15. Let $n \geq 3$ be an integer. What is the largest possible number of interior angles greater than 180° in an n-gon in the plane, given that the n-gon does not intersect itself and all its sides have the same length?

Solution
Answer: 0 if $n = 3,4$ and $n-3$ for $n \geq 5$.

If $n = 3,4$ then any such n-gon is a triangle, resp. a rhombus, therefore the answer is 0.

If $n = 5$ then consider a triangle with side lengths 2,2,1. Now move the vertex between sides of length 2 towards the opposite side by 0.0000001 units. Consider the triangle as a closed physical chain of links of length 1 that are aluminium tubes and through them is a closed rubber string. So deform the chain on a level surface so that links of the chain move towards the inside of the triangle, by fixing the vertices of the triangle to the surface. Geometrically, the links which are on sides of length 2 are now distinct chords of a large circle, attached to each other by their endpoints.

If $n > 5$ then first consider a triangle of integer sides lengths, with sides of as equal lengths as possible, so that the sum of side lengths is n. Imagine a closed aluminium chain on a rubber string, as in the previous case. Now move two of the vertices of the triangle towards the third one be 0.00000001 units each. Again consider the chain on a level surface and deform it so that its links move towards the interior of the triangle. Geometrically each side of a triangle is deformed into consecutive equal-length chords of a large circle with centre far away from the original triangle.

For $n \geq 5$ this gives an example of $n-3$ interior angles of more than 180°.

To see that $n-2$ or more such angles is not possible, note than the sum of interior angles of any n-gon (that does not cut itself) is equal to $(n-2)180°$, but already the sum of the 'large' angles would be greater than that if there were at least $n-2$ of 'large' angles of size greater than 180°.

Problem 16. Is it possible for any group of people to choose a positive integer N and assign a positive integer to each person in the group such that the product of two persons' numbers is divisible by N if and only if they are friends?

Solution
Answer: Yes, this is always possible.

Consider a graph with a vertex for each person in the group. For each pair of friends we join the corresponding vertices by a red edge. If a pair are not friends, we join their vertices with a blue edge.

Let us label blue edges with different primes $p_1, \dots, p_k$. To a vertex A we assign the number $n(A) = \frac{P^2}{P(A)}$, where $P = p_1 p_2 \dots p_k$, and $P(A)$ is the product of the primes on all blue edges starting from A (for the empty set the product of all its elements equals 1). Now take $N = P^3$.

Let us check that all conditions are satisfied. If vertices A and B are connected by a red edge, then $P(A)$ and $P(B)$ are coprime, hence $P(A)P(B) \mid P$ and $P^3 \mid n(A)n(B) = \frac{P^4}{P(A)P(B)}$. If vertices A and B are connected by a blue edge labelled with a prime q, then q^2 divides neither $n(A)$ nor $n(B)$. Hence q^3 does not divide $n(A)n(B)$.

Problem 17. Determine whether the equation

$$x^4 + y^3 = z! + 7$$

has an infinite number of solutions in positive integers.

Solution
We consider the equation modulo 13 since both 3 and 4 divides $12 = 13 - 1$. Now $x^4 \mod 13 \in \{0,1,3,9\}$ and $y^3 \mod 13 \in \{0,1,5,8,12\}$. We can verify that $x^4 + y^3 \not\equiv 7 \mod 13$. However $z! + 7 \equiv 7 \mod 13$ if $z \geq 13$, what leads to a conclusion that this equation has no solutions with $z \geq 13$, what proves that it has finite number of solutions.

Problem 18. Let $p>3$ be a prime and let $a_1,a_2,\ldots,a_{\frac{p-1}{2}}$ be a permutation of $1,2,\ldots,\frac{p-1}{2}$. For which p is it always possible to determine the sequence $a_1,a_2,\ldots,a_{\frac{p-1}{2}}$ if for all $i,j\in\{1,2,\ldots,\frac{p-1}{2}\}$ with $i\neq j$ the residue of a_ia_j modulo p is known?

Solution

Answer: For all primes $p>5$.

When $p=5$ it is clear that it is not possible to determine a_1 and a_2 from the residue of a_1a_2 modulo 5.

Assume that $p>5$. Now $\frac{p-1}{2}\geq 3$. For all $i\in\{1,2,\ldots,\frac{p-1}{2}\}$ it is possible to choose $j,k\in\{1,2,\ldots,\frac{p-1}{2}\}$ such that i, j and k are different. Thus we know

$$a_i^2\equiv(a_ia_j)(a_ia_k)(a_ja_k)^{-1}\pmod p.$$

The equation $x^2\equiv a\pmod p$ has exactly one solution in $\{1,2,\ldots,\frac{p-1}{2}\}$, and hence it is possible to determine a_i for all i.

Problem 19. For an integer $n\geq 1$ let $a(n)$ denote the total number of carries which arise when adding 2017 and $n\cdot 2017$. The first few values are given by $a(1)=1$, $a(2)=1$, $a(3)=0$, which can be seen from the following:

001	001	000
2017	4034	6051
+2017	+2017	+2017
=4034	=6051	=8068

Prove that

$$a(1)+a(2)+\ldots+a(10^{2017}-2)+a(10^{2017}-1)=10\cdot\frac{10^{2017}-1}{9}.$$

Solution

Solution 1: Let $k(m)$ be the residue of m when divided by 10^k. There is a carry at the digit representing 10^k exactly when $k(2017)+k(n\cdot 2017)>10^k$. Thus the number of 10-, 100-, 1000- and 10000-carries are, respectively,

$$\left\lfloor\frac{7\cdot 10^{2017}}{10}\right\rfloor,\quad\left\lfloor\frac{17\cdot 10^{2017}}{100}\right\rfloor,\quad\left\lfloor\frac{17\cdot 10^{2017}}{1000}\right\rfloor,\quad\left\lfloor\frac{2017\cdot 10^{2017}}{10000}\right\rfloor,$$

and similarly for the rest of the carries. Thus

$$\sum_{n=1}^{10^{2017}-1}a(n)=7\cdot 10^{2016}+17\cdot 10^{2015}+\ldots+2017+201+20+2$$

$$=(2+0+1+7)\cdot 10^{2016}+(2+0+1+7)\cdot 10^{2015}+\ldots+(2+0+1+7)=10\cdot\frac{10^{2017}-1}{9}$$

Solution 2: Let $s(n)$ denote the digit sum of n. Then we claim the following.

Lemma. We have

$$s(n+m)=s(n)+s(m)-9a(n,m),\tag{1}$$

where $a(n,m)$ denotes the total number of carries, which arises when adding n and m.

Proof: We proceed by induction on the maximal number of digits k of n and m.

If both n and m are single digit numbers then we have just two cases. If $n+m<10$, then we have no carries and clearly $s(n+m)=n+m=s(n)+s(m)$. If on the other hand $n+m=10+k\geq 10$, then

$$s(n+m)=1+k=1+(n+m-10)=s(n)+s(m)-9$$

Assume that the claim holds for all pair with at most k digits each. Let $n=n_1+a\cdot 10^{k+1}$ and $m=m_1+b\cdot 10^{k+1}$ where n_1 og m_1 are at most k digit numbers. If there is no carry at the $k+1$th digit, then $a(n,m)=a(n_1,m_1)$ and thus

$$s(n+m)=s(n_1+m_1)+a+b$$

$$=s(n_1)+a+s(m_1)+b-9a(n_1,m_1)=s(n)+s(m)-9a(n,m)$$

If there is a carry then $a(n,m)=1+a(n_1,m_1)$ and thus

$$s(n+m)=s(n_1+m_1)+a+b-9$$

$$= s(n_1) + a + s(m_1) + b - 9(a(n_1, m_1) + 1) = s(n) + s(m) - 9a(n, m)$$

This finishes the induction and we are done.

Now observe that $s(2017 \cdot 10^{2017}) = 2+1+7 = 10$. We now use (1) a total of $10^{2017} - 1$ times which yields

$$\begin{aligned} 10 = s(2017 \cdot 10^{2017}) &= s(2017 \cdot (10^{2017} - 1) + 2017) \\ &= s(2017 \cdot (10^{2017} - 1)) + s(2017) - 9 \cdot a(10^{2017} - 1) \\ &\vdots \\ &= s(2017) + s(2017) \cdot (10^{2017} - 1) - 9 \cdot \sum_{n=1}^{10^{2017}-1} a(n) \\ &= 10 \cdot 10^{2017} - 9 \cdot \sum_{n=1}^{10^{2017}-1} a(n) \end{aligned}$$

Thus we arrive at

$$\sum_{n=1}^{10^{2017}-1} a(n) = 10 \cdot \frac{10^{2017} - 1}{9}$$

Problem 20. Let S be the set of all ordered pairs (a, b) of integers with $0 < 2a < 2b < 2017$ such that $a^2 + b^2$ is a multiple of 2017. Prove that

$$\sum_{(a,b)\in S} a \;=\; \frac{1}{2} \sum_{(a,b)\in S} b\,.$$

Solution

Let $A = \{a : (a, b) \in S\}$ and $B = \{b : (a, b) \in S\}$. The claim is equivalent to

$$2\sum_{a\in A} a \;=\; \sum_{b\in B} b. \tag{2}$$

Assume that for some $x, y, z \in \{1, 2, \ldots, 1008\}$ both, $x^2 + y^2$ and $x^2 + z^2$, are multiples of 2017. By

$$(x^2 + y^2) - (x^2 + z^2) \;=\; y^2 - z^2 \;=\; (y + z)(y - z) \;\equiv\; 0 \pmod{2017},$$

$0 < y + z < 2017$ and the fact that 2017 is a prime number, it follows that $y = z$. Hence, A and B are disjoint, and there is a bijection $f : A \mapsto B$ such that for any $a \in A$ and $b \in B$ the pair (a, b) is in S if and only if $b = f(a)$.

We show that the mapping g defined by $g(a) = f(a) - a$ for $a \in A$ is a bijection from A to A. Then (2) follows by

$$\sum_{a\in A} a \;=\; \sum_{a\in A} g(a) \;=\; \sum_{a\in A} f(a) - \sum_{a\in A} a \;=\; \sum_{b\in B} b - \sum_{a\in A} a.$$

Let $a \in A$, and let $h(a) = \min\{a + f(a), 2017 - (a + f(a))\}$. Then $0 < 2h(a) < 2017$ and, by the definition of g, $0 < 2g(a) < 2017$. Furthermore,

$$g(a)^2 + h(a)^2 \;\equiv\; (a - f(a))^2 + (a + f(a))^2 \;\equiv\; 2(a^2 + f(a)^2) \;\equiv\; 0 \pmod{2017}.$$

If $a + f(a) \le 1008$, then $g(a) = f(a) - a < f(a) + a = h(a)$. If $a + f(a) > 1008$, then $g(a) = f(a) - a < (f(a) - a) + (2017 - 2f(a)) = 2017 - (a + f(a)) = h(a)$. Consequently, $g(a) \in A$ with $f(g(a)) = h(a)$.

It remains to show that g is injective. Assume that $g(a_1) = g(a_2)$ for some $a_1, a_2 \in A$, i.e.,

$$b_1 - a_1 \;=\; b_2 - a_2, \tag{3}$$

where $b_i = f(a_i)$ for $i = 1, 2$. Clearly, we also have $h(a_1) = h(a_2)$ then. If $h(a_1) = a_1 + b_1$ and $h(a_2) = a_2 + b_2$, then subtracting (3) from $a_1 + b_1 = a_2 + b_2$ gives $a_1 = a_2$. Similarly, if $h_1(a_1) = 2017 - (a_1 + b_1)$ and $h_2 = 2017 - (a_2 + b_2)$, then we obtain $a_1 = a_2$. Finally, if $h(a_1) = a_1 + b_1$ and $h_2 = 2017 - (a_2 + b_2)$, then $2(a_1 + b_2) = 2017$, a contradiction.

Remark: The proof as given above obviously works for any prime congruent to 1 modulo 4 in the place of 2017. With a little more effort, one can show that the statement is true for any positive odd n (vacuously, if n has a prime factor congruent to 3 modulo 4).