Baltic Way 2016 – Solutions

1. *Find all pairs of primes* (p, q) *such that*

$$p^3 - q^5 = (p+q)^2.$$

Solution. Assume first that neither of the numbers equals 3. Then, if $p \equiv q \bmod 3$, the left hand side is divisible by 3, but the right hand side is not. But if $p \equiv -q \bmod 3$, the left hand side is not divisible by 3, while the right hand side is. So this is not possible.

If $p = 3$, then $q^5 < 27$, which is impossible. Therefore $q = 3$, and the equation turns into $p^3 - 243 = (p+3)^2$ or

$$p(p^2 - p - 6) = 252 = 7 \cdot 36.$$

As $p > 3$ then $p^2 - p - 6$ is positive and increases with p. So the equation has at most one solution. It is easy to see that $p = 7$ is the one and $(7, 3)$ is a solution to the given equation.

2. *Prove or disprove the following hypotheses.*

a) *For all* $k \geq 2$, *each sequence of* k *consecutive positive integers contains a number that is not divisible by any prime number less than* k.

b) *For all* $k \geq 2$, *each sequence of* k *consecutive positive integers contains a number that is relatively prime to all other members of the sequence.*

Solution We give a counterexample to both claims. So neither of them is true.

For a), a counterexample is the sequence $(2, 3, 4, 5, 6, 7, 8, 9)$ of eight consecutive integers all of which are divisible by some prime less than 8.

To construct a counterexample to b), we notice that by the Chinese Remainder Theorem, there exists an integer x such that $x \equiv 0 \bmod 2$, $x \equiv 0 \bmod 5$, $x \equiv 0 \bmod 11$, $x \equiv 2 \bmod 3$, $x \equiv 5 \bmod 7$ and $x \equiv 10 \bmod 13$. The last three of these congruences mean that $x+16$ is a multiple of 3, 7, and 13. Now consider the sequence $(x, x+1, \ldots, x+16)$ of 17 consequtive integers. Of these all numbers $x + 2k$, $0 \leq k \leq 8$, are even and so have a common factor with some other. Of the remaining, $x + 1$, $x + 7$ and $x + 13$ are divisible by 3, $x + 3$ is a multiple of 13 as is $x + 16$, $x + 5$ is divisible by 5 as x, $x + 9$ is a multiple of 7 as $x + 2$, $x + 11$ a multiple of 11 as is x, and finally $x + 15$ is a multiple of 5 as is x.

Remark. The counterexample given to either hypothesis is the shortest possible. The only counterexamples of length 8 to the first hypothesis are those where numbers give remainders $2, 3, \ldots, 9$; $3, 4, \ldots, 10$; $-2, -3, \ldots, -9$; or $-3, -4, \ldots, -10$ modulo 210. The only counterexamples of length 17 to the second hypothesis are those where the numbers give remainders $2184, 2185, \ldots, 2200$ or $-2184, -2185, \ldots, -2200$ modulo 30030.

3. *For which integers* $n = 1, \ldots, 6$ *does the equation*

$$a^n + b^n = c^n + n$$

have a solution in integers?

Solution. A solution clearly exists for $n = 1, 2, 3$:

$$1^1 + 0^1 = 0^1 + 1, \qquad 1^2 + 1^2 = 0^2 + 2, \qquad 1^3 + 1^3 = (-1)^3 + 3.$$

We show that for $n = 4, 5, 6$ there is no solution.

For $n = 4$, the equation $a^4 + b^4 = c^4 + 4$ may be considered modulo 8. Since each fourth power $x^4 \equiv 0, 1 \bmod 8$, the expression $a^4 + b^4 - c^4$ can never be congruent to 4.

For $n = 5$, consider the equation $a^5 + b^5 = c^5 + 5$ modulo 11. As $x^5 \equiv 0$ or $\equiv \pm 1 \bmod 11$ (This can be seen by Fermat's Little Theorem or by direct computation), $a^5 + b^5 - c^5$ cannot be congruent to 5.

The case $n = 6$ is similarly dismissed by considering the equation modulo 13.

4. *Let n be a positive integer and let a, b, c, d be integers such that $n \mid a + b + c + d$ and $n \mid a^2 + b^2 + c^2 + d^2$. Show that*

$$n \mid a^4 + b^4 + c^4 + d^4 + 4abcd.$$

Solution. Consider the polynomial

$$w(x) = (x - a)(x - b)(x - c)(x - d) = x^4 + Ax^3 + Bx^2 + Cx + D.$$

It is clear that $w(a) = w(b) = w(c) = w(d) = 0$. By adding these values we get

$$\begin{gathered} w(a) + w(b) + w(c) + w(d) = a^4 + b^4 + c^4 + d^4 + A(a^3 + b^3 + c^3 + d^3) + \\ +B(a^2 + b^2 + c^2 + d^2) + C(a + b + c + d) + 4D = 0. \end{gathered}$$

Hence

$$\begin{gathered} a^4 + b^4 + c^4 + d^4 + 4D \\ = -A(a^3 + b^3 + c^3 + d^3) - B(a^2 + b^2 + c^2 + d^2) - C(a + b + c + d). \end{gathered}$$

Using Vieta's formulas, we can see that $D = abcd$ and $-A = a + b + c + d$. Therefore the right hand side of the equation above is divisible by n, and so is the left hand side.

5. *Let $p > 3$ be a prime such that $p \equiv 3 \pmod 4$. Given a positive integer a_0, define the sequence $a_0, a_1, \ldots$ of integers by $a_n = a_{n-1}^{2^n}$ for all $n = 1, 2, \ldots$. Prove that it is possible to choose a_0 such that the subsequence $a_N, a_{N+1}, a_{N+2}, \ldots$ is not constant modulo p for any positive integer N.*

Solution. Let p be a prime with residue 3 modulo 4 and $p > 3$. Then $p - 1 = u \cdot 2$ where $u > 1$ is odd. Choose $a_0 = 2$. The order of 2 modulo p (that is, the smallest positive integer t such that $2^t \equiv 1 \bmod p$) is a divisor of $\phi(p) = p - 1 = u \cdot 2$, but not a divisor of 2 since $1 < 2^2 < p$. Hence the order of 2 modulo p is not a power of 2. By definition we see that $a_n = a_0^{2^{1+2+\cdots+n}}$. Since the order of $a_0 = 2$ modulo p is not a power of 2, we know that

$a_n \not\equiv 1 \pmod p$ for all $n = 1, 2, 3, \ldots$. We proof the statement by contradiction. Assume there exists a positive integer N such that $a_n \equiv a_N \pmod p$ for all $n \geq N$. Let $d > 1$ be the order of a_N modulo p. Then $a_N \equiv a_n \equiv a_{n+1} = a_n^{2^{n+1}} \equiv a_N^{2^{n+1}} \pmod p$, and hence $a_N^{2^{n+1}-1} \equiv 1 \pmod p$ for all $n \geq N$. Now d divides $2^{n+1} - 1$ for all $n \geq N$, but this is a contradiction since

$$\gcd(2^{n+1} - 1, 2^{n+2} - 1) = \gcd(2^{n+1} - 1, 2^{n+2} - 1 - 2(2^{n+1} - 1)) = \gcd(2^{n+1} - 1, 1) = 1.$$

Hence there does not exist such an N.

6. *The set $\{1, 2, \ldots, 10\}$ is partitioned into three subsets A, B and C. For each subset the sum of its elements, the product of its elements and the sum of the digits of all its elements are calculated. Is it possible that A alone has the largest sum of elements, B alone has the largest product of elements, and C alone has the largest sum of digits?*

Solution. It is indeed possible. Choose $A = \{1, 9, 10\}$, $B = \{3, 7, 8\}$, $C = \{2, 4, 5, 6\}$. Then the sum of elements in A, B and C, respectively, is 20, 18 and 17, the sum of digits 11, 18 and 17, while the product of elements is 90, 168 and 240.

7. *Find all positive integers n for which*

$$3x^n + n(x + 2) - 3 \geq nx^2$$

holds for all real numbers x.

Solution. We show that the inequality holds for even n and only for them.

If n is odd, the for $x = -1$ the left hand side of the inequality equals $n - 6$ while the right hand side is n. So the inequality is not true for $x = -1$ for any odd n. So now assume that n is even. Since $|x| \geq x$, it is enough to prove $3x^n + 2n - 3 \geq nx^2 + n|x|$ for all x or equivalently that $3x^n + (2n - 3) \geq nx^2 + nx$ for $x \geq 0$. Now the AGM-inequality gives

$$2x^n + (n - 2) = x^n + x^n + 1 + \cdots + 1 \geq n \left(x^n \cdot x^n \cdot 1^{n-2}\right)^{\frac{1}{n}} = nx^2, \tag{1}$$

and similarly

$$x^n + (n - 1) \geq n \left(x^n \cdot 1^{n-1}\right)^{\frac{1}{n}} = nx. \tag{2}$$

Adding (1) and (2) yields the claim.

8. *Find all real numbers a for which there exists a non-constant function $f : \mathbb{R} \to \mathbb{R}$ satisfying the following two equations for all $x \in \mathbb{R}$:*

i) $f(ax) = a^2 f(x)$ *and*

ii) $f(f(x)) = af(x)$.

Solution. The conditions of the problem give two representations for $f(f(f(x)))$:

$$f(f(f(x))) = af(f(x)) = a^2 f(x)$$

and

$$f(f(f(x))) = f(af(x)) = a^2f(f(x)) = a^3f(x).$$

So $a^2f(x) = a^3f(x)$ for all x, and if there is an x such that $f(x) \neq 0$, the $a = 0$ or $a = 1$. Otherwise f is the constant function $f(x) = 0$ for all x. If $a = 1$, the function $f(x) = x$ satisfies the conditions. For $a = 0$, one possible solution is the function f,

$$f(x) = \begin{cases} 1 & \text{for } x < 0 \\ 0 & \text{for } x \geq 0 \end{cases}.$$

9. *Find all quadruples* $(a,\ b,\ c,\ d)$ *of real numbers that simultaneously satisfy the following equations:*

$$\begin{cases} a^3 + c^3 = 2 \\ a^2b + c^2d = 0 \\ b^3 + d^3 = 1 \\ ab^2 + cd^2 = -6 \end{cases}$$

Solution. Consider the polynomial $P(x) = (ax+b)^3 + (cx+d)^3 = (a^3+b^3)x^3 + 3(a^b + c^2d)x^2 + 3(ab^2 + cd^2)x + b^3 + d^3$. By the conditions of the problem, $P(x) = 2x^3 - 18x + 1$. Clearly $P(0) > 0$, $P(1) < 0$ and $P(3) > 0$. Thus P has three distinct zeroes. But $P(x) = 0$ implies $ax + b = -(cx + d)$ or $(a + c)x + b + d = 0$. This equation has only one solution, unless $a = -c$ and $b = -d$. But since the conditions of the problem do not allow this, we infer that the system of equations in the problem has no solution.

10. *Let* $a_{0,1},\ a_{0,2},\ \ldots,\ a_{0,\,2016}$ *be positive real numbers. For* $n \geq 0$ *and* $1 \leq k < 2016$ *set*

$$a_{n+1,k} = a_{n,k} + \frac{1}{2a_{n,k+1}} \quad \text{and} \quad a_{n+1,2016} = a_{n,2016} + \frac{1}{2a_{n,1}}.$$

Show that $\max\limits_{1\leq k\leq 2016} a_{2016,k} > 44$.

Solution. We prove

$$m_n^2 \geq n \tag{1}$$

for all n. The claim then follows from $44^2 = 1936 < 2016$. To prove (1), first notice that the inequality certainly holds for $n = 0$. Assume (1) is true for n. There is a k such that $a_{n,k} = m_n$. Also $a_{n,k+1} \leq m_n$ (or if $k = 2016$, $a_{n,1} \leq m_n$). Now (assuming $k < 2016$)

$$a_{n+1,k}^2 = \left(m_n + \frac{1}{2a_{n,k+1}}\right)^2 = m_n^2 + \frac{m_n}{a_{n,k+1}} + \frac{1}{4a_{n,k+1}^2} > n + 1.$$

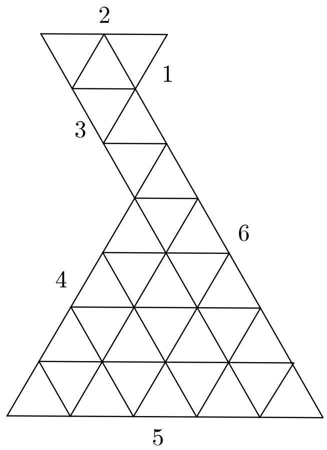
Since $m_{n+1}^2 \geq a_{n+1,\,k}^2$, we are done.

11. *The set A consists of 2016 positive integers. All prime divisors of these numbers are smaller than 30. Prove that there are four distinct numbers a, b, c and d in A such that $abcd$ is a perfect square.*

Solution There are nine prime numbers smaller than 29. Let us denote them as $p_1, p_2, ..., p_9$. To each number n from A we can assign a 9-element sequence $(n_1, n_2, ..., n_9)$ such that $n_i = 1$ when in factorization of n p_i has odd exponent, and $n_i = 0$ otherwise. There are only 512 different 9-element $\{0, 1\}$-sequences, so there exist some four numbers a, b, c and d in A that have identical sequences assigned. It is easy to see that such numbers satisfy conditions of the problem.

12. *Does there exist a hexagon (not necessarily convex) with side lengths 1, 2, 3, 4, 5, 6 (not necessarily in this order) that can be tiled with a) 31 b) 32 equilateral triangles with side length 1?*

Solution. The adjoining figure shows that question a) can be answered positively.



For a negative answer to b), we show that the number of triangles has to be odd. Assume there are x triangles in the triangulation. They hav altogether $3x$ sides. Of these, $1 + 2 + 3 + 4 + 6 = 21$ are on the perimeter of the hexagon. The remaining $3x - 21$ sides are in the interior, and they touch each other pairwise. So $3x - 21$ has to be even, which is only possible, if x is odd.

13. *Let n numbers all equal to 1 be written on a blackboard. A move consists of replacing two numbers on the board with two copies of their sum. It happens that after h moves all n numbers on the blackboard are equal to m. Prove that $h \leq \frac{1}{2}n \log_2 m$.*

Solution. Let the product of the numbers after the k-th move be a_k. Suppose the numbers involved in a move were a and b. By the arithmetic-geometric mean inequality, $(a + b)(a + b) \geq 4ab$. Therefore, regardless of the choice of the numbers in the move, $a_k \geq 4a_{k-1}$, and since $a_0 = 1$, $a_h = m^n$, we have $m^n \geq 4^h = 2^{2h}$ and $h \leq \frac{1}{2}n \log_2 m$.

14. *A cube consists of 4^3 unit cubes each containing an integer. At each move, you choose a unit cube and increase by 1 all the integers in the neighbouring cubes having a face in common with the chosen cube. Is it possible to reach a position where all the 4^3 integers are divisible by 3, no matter what the starting position is?*

Solution. Two unit cubes with a common face are called neighbours. Colour the cubes either black or white in such a way that two neighbours always have different colours. Notice that the integers in the white cubes only change when a black cube is chosen. Now recolour the white cubes that have exactly 4 neighbours and make them green. If we look at a random black cube it has either 0, 3 or 6 white neighbours. Hence if we look at the sum of the integers in the white cubes, it changes by 0, 3 or 6 in each turn. From this it follows that if this sum is not divisible by 3 at the beginning, it will never be, and none of the integers in the white cubes is divisible by 3 at any state.

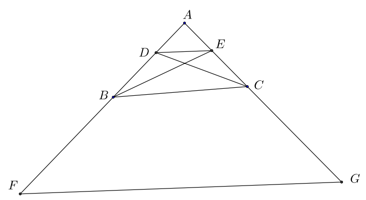
15. *The Baltic Sea has 2016 harbours. There are two-way ferry connections between some of them. It is impossible to make a sequence of direct voyages $C_1 - C_2 - \cdots - C_{1062}$ where all the harbours $C_1, \ldots, C_{1062}$ are distinct. Prove that there exist two disjoint sets A and B of 477 harbours each, such that there is no harbour in A with a direct ferry connection to a harbour in B.*

Solution. Let V be the set of all harbours. Take any harbour C_1 and set $U = V \setminus \{C_1\}$, $W = \emptyset$. If there is a ferry connection from C to another harbour, say C_2 in V, consider the route C_1C_2 and remove C_2 from U. Extend it as long as possible. Since there is no route of length 1061, So we have a route from C_1 to some C_k, $k \leq 1061$, and no connection from C_k to a harbor not already included in the route exists. There are at least $2016 - 1062$ harbours in U. Now we move C_k from U to W and try to extend the route from C_{k-1} onwards. The extension again terminates at some harbor, which we then move from U to W. If no connection from C_1 to any harbour exists, we move C_1 to W and start the process again from some other harbour. This algorithm produces two sets of harbours, W and U, between which there are no direct connections. During the process, the number of harbours in U always decreases by 1 and the number of harbours in W increases by 1. So at some point the number of harbours is the same, and it then is at least $\frac{1}{2}(2016-1062) = 477$. By removing, if necessary, some harbours fron U and W we get sets of exactly 477 harbours.

16. *In triangle ABC, the points D and E are the intersections of the angular bisectors from C and B with the sides AB and AC, respectively. Points F and G on the extensions of AB and AC beyond B and C, respectively, satisfy $BF = CG = BC$. Prove that $FG \parallel DE$.*

Solution. Since BE and CD are angle bisectors,

$$\frac{AD}{AB} = \frac{AC}{AC + BC}, \quad \frac{AE}{AC} = \frac{AB}{AB + BC}.$$



So

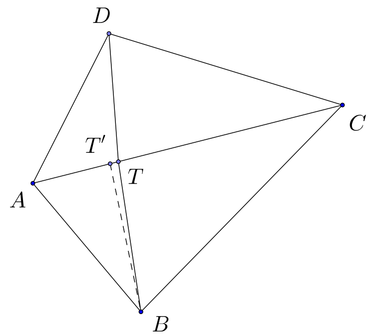
$$\frac{AD}{AF} = \frac{AD}{AB} \cdot \frac{AB}{AF} = \frac{AC \cdot AB}{(AC + BC)(AB + BC)}$$

and

$$\frac{AE}{AG} = \frac{AE}{AC} \cdot \frac{AC}{AG} = \frac{AB \cdot AC}{(AB + AC)(AC + BC)}.$$

Since $\frac{AD}{AF} = \frac{AE}{AG}$, DE and FG are parallel.

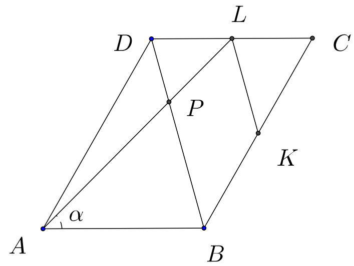
17. *Let $ABCD$ be a convex quadrilateral with $AB = AD$. Let T be a point on the diagonal AC such that $\angle ABT + \angle ADT = \angle BCD$. Prove that $AT + AC \geq AB + AD$.*



Solution. On the segment AC, consider the unique point T' such that $AT' \cdot AC = AB^2$. The triangles ABC and $AT'B$ are similar: they have the angle at A common, and $AT' : AB = AB : AC$. So $\angle ABT' = \angle ACB$. Analogously, $\angle ADT' = \angle ACD$. So $\angle ABT' + \angle ADT' = \angle BCD$. But $ABT' + ADT'$ increases strictly monotonously, as T' moves from A towards C on AC. The assumption on T implies that $T' = T$. So, by the arithmetic-geometric mean inequality,

$$AB + AD = 2AB = 2\sqrt{AT \cdot AC} \leq AT + AC.$$

18. *Let $ABCD$ be a parallelogram such that $\angle BAD = 60°$. Let K and L be the midpoints of BC and CD, respectively. Assuming that $ABKL$ is a cyclic quadrilateral, find $\angle ABD$.*



Solution. Let $\angle BAL = \alpha$. Since $ABKL$ is cyclic, $\angle KKC = \alpha$. Because $LK \| DB$ and $AB \| DC$, we further have $\angle DBC = \alpha$ and $\angle ADB = \alpha$. Let BD and AL intersect at P. The triangles ABP and DBA have two equal angles, and hence $ABP \sim DBA$. So

$$\frac{AB}{DB} = \frac{BP}{AB}. \tag{1}$$

The triangles ABP and LDP are clearly similar with similarity ratio $2 : 1$. Hence $BP = \frac{2}{3}DB$. Inserting this into (1) we get

$$AB = \sqrt{\frac{2}{3}} \cdot DB.$$

The sine theorem applied to ABD (recall that $\angle DAB = 60°$) immediately gives

$$\sin\alpha = \frac{AB}{BD}\sin 60° = \sqrt{\frac{2}{3}} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{2}}{2} = \sin 45°.$$

So $\angle ABD = 180° - 60° - 45° = 75°$.

19. *Consider triangles in the plane where each vertex has integer coordinates. Such a triangle can be* legally transformed *by moving one vertex parallel to the opposite side to a different point with integer coordinates. Show that if two triangles have the same area, then there exists a series of legal transformations that transforms one to the other.*

Solution. We will first show that any such triangle can be transformed to a *special* triangle whose vertices are at (0, 0), (0, 1) and (n, 0). Since every transformation preserves the triangle's area, triangles with the same area will have the same value for n.

Define th y-*span* of a triangle to be the difference between the largest and the smallest y coordinate of its vertices. First we show that a triangle with a y-span greater than one can be transformed to a triangle with a strictly lower y-span.

Assume A has the highest and C the lowest y coordinate of ABC. Shifting C to C' by the vector $\overrightarrow{BA}$ results in the new triangle ABC' where C' has larger y coordinate than C baut lower than A, and C' has integer coordinates. If AC is parallel to the x-axis, a horizontal shift of B can be made to transform ABC into $AB'C$ where $B'C$ is vertical, and then A can be vertically shifted so that the y coordinate of A is between those of B' and C. Then the y-span of $AB'C$ can be reduced in the manner described above. Continuing the process, one necessarily arrives at a triangle with y-span equal to 1. Such a triangle then necessarily has one side, say AC, horizontal. A legal horizontal move can take B to the a position B' where AB' is horizontal and C has the highest x-coordinate. If B' is above AC, perform a vertical and a horizontal legal move to take B' to the origin; the result is a special triangle. If B' is below AC, legal transformation again can bring B' to the origin, and a final horizontal transformation of one vertex produces the desired special triangle.

The inverse of a legal transformation is again a legal transformation. Hence any two triangles having vertices with integer coordinates and same area can be legally transformed into each other via a special triangle.

20. *Let $ABCD$ be a cyclic quadrilateral with AB and CD not parallel. Let M be the midpoint of CD. Let P be a point inside $ABCD$ such that $PA = PB = CM$. Prove that AB, CD and the perpendicular bisector of MP are concurrent.*

Solution. Let ω be the circumcircle of $ABCD$. Let AB and CD intersect at X. Let ω_1 and ω_2 be the circles with centers P and M and with equal radius $PB = MC = r$. The power of X with respect to ω and ω_1 equals $XA \cdot XB$ and with respect to ω and ω_2 $XD \cdot XC$. The latter power also equals $(XM + r)(XM - r) = XM^2 - r^2$. Analogously, the first power is $XP^2 - r^2$. But since $XA \cdot XB = XD \cdot XC$, we must have $XM^2 = XP^2$ or $XM = XP$. X indeed is on the perpendicular bisector of PM, and we are done.